Large amplitude dynamics of beam subjected to a compressive axial force resting on non -linear elastic foundation

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Abstract

In this paper, the large amplitude of free vibration and buckling of Euler–Bernoulli beam rests on a non-linear elastic foundation subjected to an axial force are studied. Hamilton's principle is followed to derive governing equation of the beam response. Using an analytical method based on the Galerkin technique, the nonlinear governing equations of motion was simplified to a time-dependent Duffing equation with cubic nonlinearities and then solved using Laplace Iteration Method. Comparison between results of the present work and those available in literature review shows reasonable agreement of this method. Effects of vibration amplitude, elastic coefficients of foundation and axial force on the non-linear natural frequencies and buckling load of beams are presented. Results reveal that decreasing linear and shear parameters and increasing nonlinear parameters of foundation lead to increasing frequency ratio and buckling load ratio. Furthermore, increasing axial force decreases absolute values of both linear and nonlinear frequencies as well as natural frequency ratio.

Keywords:

Nonlinear Vibration; Buckling; Euler-Bernoulli beam; Non-linear elastic foundation; Galerkin method ; Laplace Iteration Method;

السعة الديناميكية الكبيرة لعارضة معرضة لقوة ضغط محورية مسندة على أساس مرن غير خطي

الخلاصة

في هذا البحث تم دراسة السعة العالية للاهتزاز الحر والانبعاج لعارضة اويلر-برنولي مستقرة على اساس مرن غير خطي معرض الى قوة محورية. تم استخدم مبدا هاملتون في اشتقاق المعادلة الحاكمة لاستجابة العارضة. استخدم الأسلوب التحليلي استنادا إلى تقنية Galerkin ، تم تبسيط المعادلات الغير خطية الحاكمة للحركة الى معادلة دفنج المعتمدة على الزمن من الدرجة الثالثة والغير خطية وبعد ذلك تم حلها باستخدام طريقة

لابلاس للتكرار. مقارنة النتائج التي تم الحصول عليها في هذه الدراسة و تلك المتاحة في الأدبيات اظهرت توافق معقول لهذه الطريقة . تاثيرات كل من سعة الاهتزاز ، معامل مرونة الاساس والقوة المحورية على الترددات الطبيعية اللاخطية وحمل الانبعاج للعارضة تمت دراستها. النتائج بينت بأن خفض المعاملات الخطية و معاملات القص وزيادة المعاملات اللا الخطية للأسس ادت إلى زيادة نسبة التردد و نسبة الانبعاج. بالاضافة الى ذلك، زيادة القوة المحورية قللت القيم المطلقة لكل من الترددات الخطية وغير الخطية وكذلك نسبة التردد الطبيعي.

كلمات مرشدة

الاهتزازالغير الخطي؛ الانبعاج؛عارضة اويلر-برنولي ؛ اساس لا خطي المرونة ؛ طريقة كارلكن؛ طريقة لابلاس التكرارية.

1-Introduction

Beam is one of the important mechanical elements and has numerous applications in different fields of engineering and industries such as civil, marine and aerospace structures or vehicles.

In most applications, they are subjected to non-linear vibrations which lead to material fatigue and structural damage due to increment of the oscillation amplitude [1]. Therefore, it is necessary and very important to study dynamic nonlinear behavior and natural responses of these structures at large amplitudes. Many investigations have been reported on the linear, non-linear vibration and buckling of Euler-Bernoulli beams with and without an elastic foundation. Lai et al. [2] utilized Adomian Decomposition Method to obtain the natural frequencies and mode shapes for the Euler- Bernoulli beam under various supporting conditions. Non-linear vibration behavior Euler-Bernoulli beams subjected to axial load using homotopy analysis method (HAM) is investigated by Pirbodaghi et al. [3]. Barari et al. [4] studied non-linear vibration behavior of geometrically non-linear Euler-Bernoulli beams using variational iteration method and parameter perturbation method. Non-linear vibration Euler-Bernoulli beams subjected to axial load using He's variational approach (VA) and Laplace iteration methods (LIM) is studied by Bagheri et al. [5]. Analysis of the nonlinear free vibration of simply-supported and clamped-clamped Euler-Bernoulli beams fixed at one end subjected to the axial force using Hamiltonian approach is presented by Bayat et al. [6]. Nonlinear vibration analysis of isotropic beams with simple end conditions has been investigated by Kargarnovin and Jafari [7]. They have used the HAM to obtain closed-form solutions for natural frequencies and beam deflection. Liu and Gurram [8] investigated the free vibration of Euler-Bernoulli beam

under various supporting conditions using a variational iteration method. Nonlinear analysis for simply supported beam resting on a two-parameter elastic foundation is presented by Hui-Shen [9]. The large amplitude free vibration of uniform beams on Pasternak foundation using the conservation of total energy principle is presented by Venkateswara [10]. The LIM method which was introduced by Rafieipour et al. [11] is a very powerful method in solving non-linear differential equations and its effectiveness is proved in studying non-linear vibration of composite plate. It was shown that this is one of the best analytical methods due to the rate of convergence and its accuracy.

The objectives of the present paper are to use the Laplace Iteration Method to obtain approximate analytical solutions for large amplitude dynamic of Euler–Bernoulli beam subjected to a compressive axial force resting on non -linear elastic foundation and to study the influence of different parameters on the frequency and post-buckling loads of the beam.

2. Theoretical formulation

A simply supported beam with uniform cross section made of a homogenous isotropic material with negligible damping is considered. The beam is supported on an elastic foundation with cubic nonlinearity and shearing layer as shown in Fig.1. The beam is modeled according to Euler Bernoulli beam theory. Planes of the cross sections remain planes after deformation, straight lines normal to the midplane of the beam remain normal, and straight lines in the transverse direction of the cross section do not change length, [4]

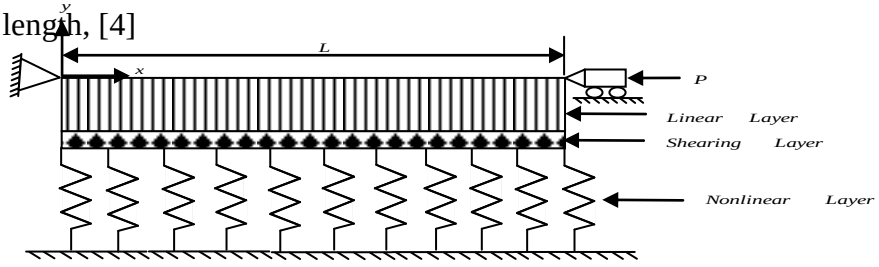

Fig. 1: A schematic of an Euler-Bernoulli beam subjected to an axial load resting on non -linear elastic foundation

Based on the Euler–Bernoulli beam theory, the nonlinear strain–displacement relations of the beam with the axially immovable ends are given by, [14]

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 , \qquad k_x = \frac{\partial^2 w}{\partial x^2} \qquad -----(1)$$

where u is the longitudinal displacement, w is the lateral displacement and k_x is the curvature of the beam, respectively. In this study, the equations of motion are derived by using Hamilton's principle. This principle can be expressed as, [12]

$$\delta \int_0^t [K_e - (U_s - W)]\, dt = 0 \qquad -----(3)$$

In equation (5), K_e is the kinetic energy, U_s is the elastic strain energy and $W_{ext.}$ is the work done by the external applied forces and these are given by, [10,12]

$$T_e = \frac{1}{2}\int_0^L m\left\{\frac{\partial w}{\partial t}\right\}^2 dx$$

$$U_s = \frac{1}{2}\int_0^L \left(EA\left\{\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)\right\}^2 + EI\left\{\frac{\partial^2 w}{\partial x^2}\right\}^2\right) dx \qquad -----(4)$$

$$W_{ext.} = \frac{1}{2}\int_0^L k_L w^2 dx + \frac{1}{2}\int_0^L k_{NL} w^4 dx - \frac{1}{2}\int_0^L k_S \left(\frac{\partial w}{\partial x}\right)^2 dx + \frac{1}{2}\int_0^L p\left(\frac{\partial w}{\partial x}\right)^2 dx$$

Applying the Extended Hamilton's principle [12], the governing equation of transverse vibration of beam including axial stretching on a nonlinear elastic foundation can be obtained as:

$$\frac{\partial^2}{\partial x}\left\{EI\frac{\partial^2 w}{\partial x^2}\right\} + p\frac{\partial^2 w}{\partial x^2} - \left\{\frac{EA}{2L}\int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx\right\}\frac{\partial^2 w}{\partial x^2} + m\frac{\partial^2 w}{\partial \bar{t}^2} + k_L w + k_{NL} w^3 - k_S\left(\frac{\partial^2 w}{\partial x^2}\right) = 0 \qquad -----(5)$$

For convenience, the following non-dimensional variables are used:

$$X = x/L\,,\ W = w/r\,,\ t = \bar{t}\,(EI/mL^4)^{1/2}\,,\ r = \sqrt{(I/A)}\,, \qquad ------(6)$$

As a result equation (5) can be written as follows:

$$\frac{\partial^2 W}{\partial t^2} + \frac{\partial^4 W}{\partial X^4} + P\frac{\partial^2 W}{\partial X^2} - \frac{1}{2}\frac{\partial^2 W}{\partial X^2}\int_0^l \left(\frac{\partial W}{\partial X}\right)^2 dx + K_L W + K_{NL} W^3 - K_S\left(\frac{\partial^2 W}{\partial X^2}\right) = 0 \qquad -----(7)$$

In which:

$$P = pL^2/EI\,, \qquad K_L = kL^4/EI \quad, \quad K_{NL} = k_{NL}L^6/EI\,, \qquad K_S = k_S L^2/EI$$

$$-----(8)$$

Assuming $W(X,t) = q(t)\phi(x)$ where $\phi(x)$ is the first normal mode of the beam [16] and using the Galerkin method, the governing equation of motion is obtained as follows:

$$\frac{d^2 q(t)}{dt^2} + (\alpha_1 + P\alpha_P + \alpha_{K_L} + \alpha_{K_S})q(t) + (\alpha_{K_{NL}} + \alpha_3)q^3(t) = 0 \qquad ------(10)$$

where coefficients α_i are presented in Appendix A. The beam centroid is subjected to the following initial conditions, [3]

$$q(x,0) = W_{\max} \quad \frac{dq(x,0)}{dt} = 0 \qquad ------(11)$$

From equation (10), the Post-buckling load–deflection relation of the beam can be obtained as:

$$P_{NL} = \frac{-(\alpha_1 + \alpha_{K_L} + \alpha_{K_S}) - (\alpha_{K_{NL}} + \alpha_3) q(t)^2}{\alpha_2} \qquad ------(12)$$

It should be noted that neglecting the contribution of q in equation (16), the linear buckling load can be determined as, [17]

$$P_L = P_{cr} = \frac{-(\alpha_1 + \alpha_{K_L} + \alpha_{K_S})}{\alpha_2} \qquad ------(13)$$

Equation (10) is strongly nonlinear and nobody can find an exact analytical closed form solution for $q(t)$ and ω_{NL}. Although numerical methods can be implemented to get over this problem but, they cannot offer any suitable way for parametric study. Therefore, it will be valuable if a powerful analytical approximate method exists that presents an accurate approximation of $q(t)$ and ω_{NL} while providing the ability to parametric study of the problem.

3- Description of the proposed method

Using the Laplace Transformation method, an analytical approximated technique is proposed to present an accurate solution for nonlinear differential equations. To clarify the basic ideas of proposed method consider the following second order differential equation [11],

$$\ddot{u}(t) + N\{u(t)\} = 0 \qquad -----(14)$$

With artificial zero initial conditions and N is a nonlinear operator. Adding and subtracting the term $\omega^2 u(t)$, equation (14) can be written in the form

$$\ddot{u}(t) + \omega^2 u(t) = L\{u(t)\} = \omega^2 u(t) - N\{u(t)\} \qquad -----(15)$$

where L is a linear operator. After taking Laplace transform of both sides of the equation (15) in the usual way and then implementing the Laplace inverse transform by using the Convolution theorem yields:

$$u(t) = \int_0^t (\omega^2 u(\tau) - N\{u(\tau)\}) \frac{1}{\omega} \sin(\omega(t-\tau))\, d\tau \qquad -----(16)$$

Now, the actual initial conditions must be imposed. Finally the following iteration formulation

can be used, [13]

$$u_{n+1} = u_0 + \frac{1}{\omega} \int_0^t (\omega^2 u_n(\tau) - N\{u_n(\tau)\}) \sin(\omega(t-\tau))\, d\tau \qquad -----(17)$$

Knowing the initial approximation u_0 , the next approximations, u_n, $n > 0$ can be determined

from previous iterations. Consequently, the exact solution may be obtained by using

$$u = \lim_{n \to \infty} u_n \qquad -----(18)$$

The method proposed here can be applied in various non-linear problems. However, there is no

need for any linearization and any small parameter, also, the obtained approximate solutions converge quickly to the exact one.

4-Implementation of the proposed method

Rewriting equation (10) in the standard form of equation (15) results in the following equation:

$$\frac{d^2\eta(t)}{dt^2} + \omega^2 \eta(t) = \omega^2 \eta(t) - N\{\eta(t)\} \qquad ----(19)$$

where

$$N\{\eta(t)\} = \alpha\eta(t) + \beta\eta^3(t) \quad \text{and} \quad \alpha = \alpha_1 + P\alpha_P + \alpha_{K_L} + \alpha_{K_S}, \quad \beta = (\alpha_{K_{NL}} + \alpha_3) \qquad ----(20)$$

Applying the proposed method, the following iterative formula is formed as:

$$\eta_{n+1} = \eta_0 + \frac{1}{\omega} \int_0^t (\omega^2 (\eta_n(\tau)) - N\{\eta_n(\tau)\}) \sin(\omega(t-\tau))\, d\tau \qquad ----(21)$$

Equation(19) will be homogeneous, if left side of this equation is considered to be zero. So, it's homogeneous Solution

$$\eta_0(t) = W_{\max} \cos(\omega t) \qquad ----(22)$$

is considered as the zero approximation for using in iterative equation (21)

Expanding $f(\eta_0(\tau))$ we have:

$$f(\eta_o(\tau)) = \left(-\alpha W_{max} + \omega^2 W_{max} - \frac{3}{4}\beta W_{max}^3 \right)\cos(\omega t) - \frac{1}{4}\beta W_{max}^3 \cos(3\omega t) \quad ----(23)$$

Considering the relation:

$$\frac{1}{\omega}\int_0^t (\cos(m\omega\tau))\sin(\omega(t-\tau))d\tau = \begin{cases} \dfrac{\cos(\omega t) - \cos(m\omega t)}{\omega^2(m^2-1)} & m \neq 1 \\ \dfrac{t\sin(\omega t)}{2\omega} & m = 1 \end{cases} \quad (24)$$

The coefficient of the term $\cos(\omega t)$ in $f(\eta_0(\tau))$ should be vanished in order to avoid secular terms in subsequent iterations.

As a result, the nonlinear natural frequency of the motion can be expressed as:

$$\omega_{LIM} = \sqrt{\alpha + \frac{3}{4}\beta W_{max}^2} \quad ----(25)$$

Then, the nonlinear to linear frequency ration can be determined as:

$$\frac{\omega_{NL}}{\omega_L} = \frac{\sqrt{\alpha + \frac{3}{4}\beta W_{ma}^2}}{\sqrt{\alpha}} \quad -----(26)$$

and, the zero-order approximate solution can be easily determined as:

$$\eta(t) = W_{max}\cos\left(\sqrt{\alpha + \frac{3}{4}\beta W_{max}^2}\, t\right) \quad ----(27)$$

5-Results and discussion

In order to demonstrate the accuracy and effectiveness of the LIM, The procedure explained in previous section is applied to simply supported and clamped beams. Table 1 shows the comparison of non-linear to linear frequency ratio (ω_{NL}/ω_L) with those reported in the literature. It can be observed that there is a reasonable agreement between the results obtained from the LIM and those reported by Ref's [14 and 15]. By increasing the amplitude of vibration the difference between the non-linear frequency and linear frequency increases. In general, large vibration amplitude will yield a higher frequency

Table 1 Comparison of nonlinear to linear frequency ratio (ω_{NL}/ω_L)

Boundary condition	W_{max}	Ref. [14]	Ref. [15]	Present Study
Simply supported	1	1.0897	1.0891	1.0897
	2	1.3229	1.3228	1.3228
	3	1.6394	1.6256	1.6293
Clamped-clamped	1	1.0552	1.0572	1.0551
	2	1.2056	1.2125	1.2056
	3	1.3904	1.4344	1.4214

The exact nonlinear frequency (ω_{ex}) of Duffing equation (10) can be determined as [18]:

$$\omega_{ex} = 2\pi\left(4\int_0^{\pi/2} \frac{d\tau}{\sqrt{(\alpha_1 + P\alpha_2 + \alpha_{K_L} + \alpha_{K_S}) + \frac{1}{2}(\alpha_{K_{NL}} + \alpha_3)W_{max}^2(1+\sin^2(\tau))}}\right)^{-1} \quad ----(28)$$

The another verification attempt, frequency ration obtained by equation (25) for versus dimensionless amplitude is compared with the results obtained from exact frequency ratio using equation (28), as shown in Table (2). A reasonably good agreement with exact solution for nonlinear analysis of beam can be observed.

Table 2: Comparison of nonlinear to linear frequency ratio (ω_{NL}/ω_L)

a	Present Study	Exact	Relative Error
0	1.0000	1.0000	0.000
1	1.0233	1.0213	0.002
2	1.0882	1.0863	0.019
3	1.1891	1.1871	0.002
4	1.3175	1.3167	0.008
5	1.4662	1.4658	0.004

The effect of the elastic foundation coefficients and axial force on the nonlinear natural frequency and post-buckling behavior of simply supported beams is studied. The figures (2-5) demonstrate effects of foundation parameters, linear (K_L), shear (K_S) and nonlinear (K_{NL}) for simply supported beam. In all figures, the dimensionless frequency ratio (ω_{NL}/ω_L) and dimensionless nonlinear to linear buckling load ratio (P_{NL}/P_L) versus dimensionless amplitude are presented. It is worth mentioning that (P_{NL}) is determined by maximizing equation (12) in one period

of vibration. It can be concluded from figures (2, 3) together with Table (3) that an increase in the value of linear elastic foundation stiffness and shearing layer stiffness results in decreasing hardening characteristic of the beam i.e. decrease in the rate of increase in the nonlinear frequency and post-buckling strength with amplitude. Whereas it is shown in figure (4) that an increase in the value of nonlinear elastic foundation stiffness has inverse results on the nonlinear frequency and post-buckling strength i.e. the rate of increase in these figures enhances with an increase in nonlinear foundation stiffness. The results might be explained by writing frequency ratio based on equation (26) as:

$$\frac{\omega_{NL}}{\omega_L} = \sqrt{1 + \frac{3}{4}\frac{\beta}{\alpha} W_{max}^{\ 2}} \qquad ------(28)$$

Considering the fact that α is a function on both linear and shear stiffness of the foundation while β remain constant. This yields to decreasing frequency ration by increasing K_L and K_S. On the other hand, however the effect of nonlinear stiffness coefficient of the foundation appears only in β which results in increasing the frequenacy ratio.

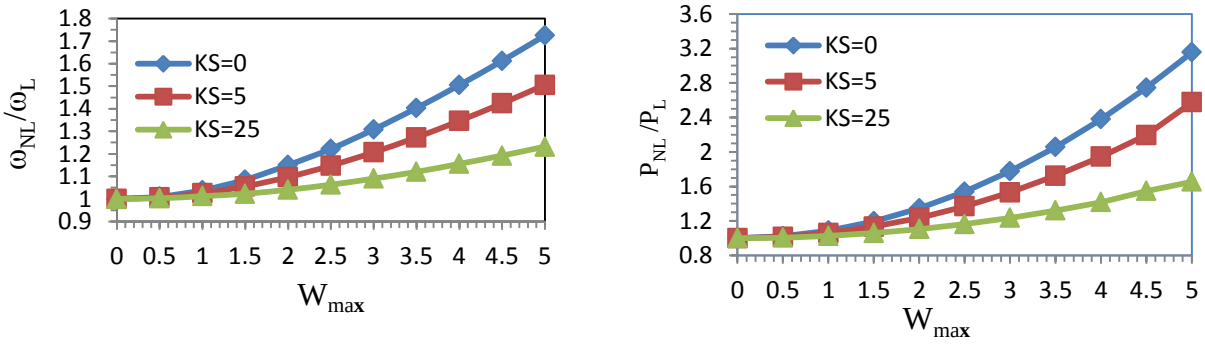

Fig.2: Effects of the shear foundation stiffness on the frequency and post-buckling load-deflection.

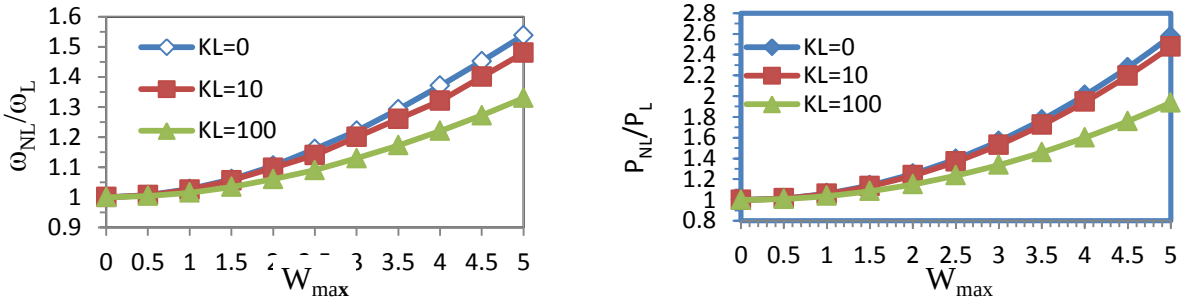

Fig.3: Effects of the linear foundation stiffness on the frequency and post-buckling load-deflection.

Table 3: Comparison of nonlinear frequency (ω_{NL}) and nonlinear to linear frequency ratio (ω_{NL}/ω_L) with change of different factors for beam, a=2.

Nonlinear natural frequency						Nonlinear to linear frequency			
P	K_L	K_{NL}				K_{NL}			
		0		100		0		100	
		K_S		K_S		K_S		K_S	
		0	25	0	25	0	25	0	25
0	0	10.227	18.749	17.564	23.564	1.0373	1.0106	1.779	1.2702
	50	12.447	20.038	18.935	24.602	1.0277	1.009	1.559	1.239
5	0	5.609	17.383	16.099	22.493	1.0742	1.0124	2.322	1.31
	50	10.269	18.767	17.583	23.577	1.0368	1.0106	1.776	1.269

The effect of initial and pre-buckled non-dimensional axial force on the nonlinear natural frequency is shown in figure (5). It can be concluded that the values of nonlinear frequency decrease with increase in the axial force, whereas nonlinear to linear frequency ratio enhances with an increase in the axial force value.

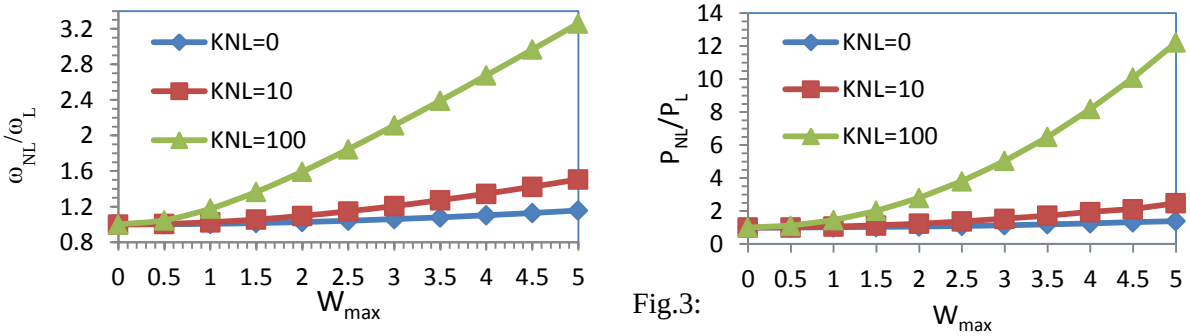

Fig.3: Effects of the non- linear foundation stiffness on the frequency and post-buckling load-deflection.

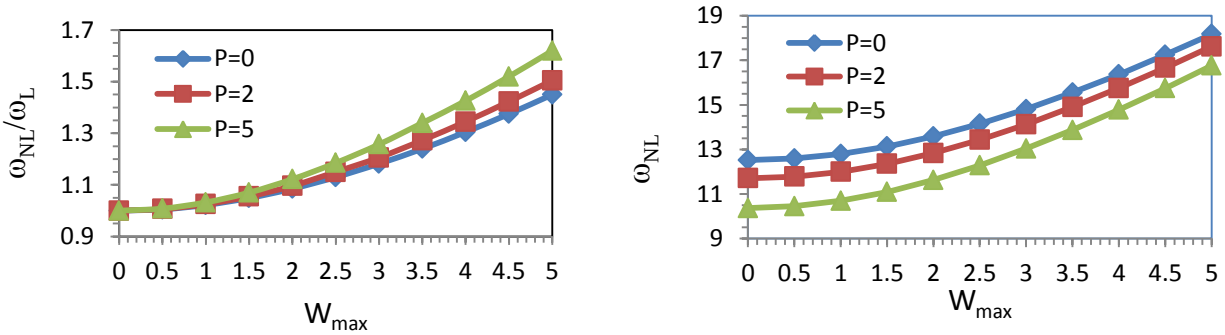

Fig.5: Effects of the axial load on the frequency ratio of the beam

6- Conclusions

Laplace Iteration Method is used to obtain the analytical expressions for the nonlinear free vibration of an Euler-Bernoulli beam on a nonlinear elastic foundation subjected to axial compressive force. The influence of foundation stiffness parameters and initial pre-buckled axial load on the nonlinear natural frequency and post-buckling load-deflection has been investigated. Based on a detailed study, the following observations are made:

1- The influence of linear and shear layer stiffness parameters is to decrease the nonlinear behavior of the beam, whereas the nonlinear layer stiffness of the foundation increases the strength of the nonlinearity.

2- The rate of increase in nonlinear to linear natural frequency and post-buckling to critical load ratios is very low at small amplitudes. However, as the amplitude increases, the effect of nonlinearity on these parameters becomes significant.

4-The nonlinear to linear frequency ratio enhances with an increase in the value of the axial force, whereas the values of nonlinear frequency decrease.

5-Laplace Iteration Method provides the ability for parametric study of the considered problem. Results revealed that the presented method offers accurate solution with low computational effort.

6- Comparison between the results of the present study and other methods available in the literature shows the accuracy of the method.

Appendix A:

$$\alpha_1 = \frac{\int_0^l \phi''''\phi dx}{\int_0^l \phi^2 dx} \quad , \quad \alpha_2 = \frac{\int_0^l \phi''\phi dx}{\int_0^l \phi^2 dx} \quad , \quad \alpha_{K_S} = -K_S \frac{\int_0^l \phi''\phi dx}{\int_0^l \phi^2 dx}$$

$$\alpha_{K_{NL}} = K_{NL} \frac{\int_0^l \phi^4 \phi dx}{\int_0^l \phi^2 dx} \quad , \quad \alpha_3 = -0.5 \frac{\int_0^l \phi''\phi dx \int_0^l \phi'^2 dx}{\int_0^l \phi^2 dx} \quad , \quad \alpha_{K_L} = K_L$$

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Nomenclature

A Cross section of beam (m^2)

E Young's modulus (N/m^2)

I Second moment of area (m^4)

$\bar{k}_L$ Linear foundation stiffness (N/m)

Greek symbols

ω_L Linear fundamental frequency (rad/s)

ω_{NL} Non-linear frequency (rad/s)

$\phi(x)$ Trial function

$\bar{k}_{NL}$	Nonlinear foundation stiffness (N/m^3)
$\bar{k}_S$	Shearing layer stiffness (Nm)
k_x	The curvature of the beam
ε_x	The nonlinear axial strain
L	Length of beam (m)
m	Mass per unit length (kg/m)
p	Axial load (N)
r	Radius of gyration of the cross section (m)
$\bar{t}$	Time (s)
u	Longitudinal displacement (m)
w	Transverse displacement (m)
$W(t)$	Time-dependent deflection parameter
W_{max}	Dimensionless maximum amplitude of oscillation (m)